Hypothesis on the geometry of a multidimensional universe

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Abstract. Multidimensional cosmology typically uses the following geometry: $ds^2=g_{\mu\nu}dx^\mu dx^\nu+e^{2\phi}h_{mn}dy^m dy^n$. This position does not come from primary principles: rather it must be considered as a generalization of the Friedmann metric. Besides, cosmologies inspired by unification models often do not allow geometries of this kind without losing stability. In the present work, we discuss about different metric beginning with the hypothesis that the ordinary space-time can be locally regarded as a pseudo-Riemannian hypersurface isometrically embedded in M_n. Furthermore, we demonstrate that this geometry is capable to resolve, in a simple way, the non-linear multidimensional Einstein-Gauss-Bonnet model. Finally, we show how said geometry can contribute to the establishing of the cosmological extended principle needed to understand the nature of a multidimensional universe.

Introduction. Cosmological models are derived from Einstein equations through the geometrical hypothesis contained in the Cosmological Principle[1]. On the contrary, multidimensional models, tipically inspired from unification theories, are derived from dimensionally extended Einstein equations after using a generalization of the Friedmann metric[2÷12]: this, assuming a block-diagonal form, gives to the internal space a scale factor depending from the external coordinates.

Practically, there are not strong geometric foundations or a cosmological extended principle able to support this metric ansatz: obviously this position would find a later justification should it be able in a position to attain the desired cosmological solutions.

However, in linear models Kasner-type solutions[13,15] exist able to anisotropize by cosmological evolution the multidimensional space confining the internal space to distances of the Planck length order. These solutions can turn out to be definitively stable if matter contributions are considered[3,4,7,14,15,17]. Unfortunately, stability does not seem possible if the gravitational lagrangian contains the non-linear Gauss-Bonnet term[12]: on the other hand, said term is present in the promising unification theories obtained from the superstrings heterotic model[18,19,21,31,36].

It is worth mentioning that the generalization of the Friedmann metric is historically linked to the Kaluza-Klein unification theories[22÷28]: in these theories, the off-diagonal elements of the multidimensional metric tensor generate, through the isometries of the internal space, the desired non-abelian gauge fields. In this way, the multidimensional metric turns out to be block-diagonal if vacuum models are considered, which means, in other words, that the vacuum expectation value takes the place of the gauge potentials. This allows to regard the multidimensional space-time as a direct product of two subspaces and therefore to consistently use the generalization of the Friedmann metric. This approach to unification is valid in supergravity theories[23,27,28,29] where it is the multidimensional metric tensor that produces the gauge fields[16,29]; on the contrary, in non-linear *10*-dimensional cosmologies inspired to the superstrings heterotic model the gauge fields are automatically present: in fact these fields are, as the graviton, other bosonic massless states of the *10*-dimensional heterotic superstring[18,28,30,31,36]. So, in this case, it is not enough clear how and through which hypotesis it is possible to support the idea of a *10*-dimensional universe given by a direct product of two subspaces.

The present work is dedicated to the investigation of the geometric hypotesis relating to a universe presumably multidimensional, trying to express an extended cosmological principle. The work is organized as follows: in the next section the form of the multidimensional metric, rather than assumed, is obtained; further examination is than given to the comparison between this obtained metric and the generalized Friedmann metric. In the following section this metric is used to solve the non-linear Einstein-Gauss-Bonnet model. Finally, in the last section, the embedding geometry is inductively used to try to express an extended cosmological principle that, though containing the traditional Cosmological Principle, turns out to be consistent with the hypotesis relating to the multidimensionality of the universe.

The geometry. Let S_4 be the ordinary space-time and $g_{\mu\nu}$ be its metric: if we assume that the universe got a multidimensional structure, we can locally consider S_4 as an ipersurface isometrically embedded in a *n*-dimensional Minkowski space-time. The choice of M_n is essentially due to the fact that the embedding in a flat space-time is quite simple[32]; moreover, the unification models which we refer to, assume M_n as the ground state[28]. As we will see, this choice does not provide trivial results; on the contrary, it allows a generalization of tangent space to S_4.

The embedding is then realized when a point of S_4, with coordinates x^λ $(..,\lambda,..=0,..,3)$, can be described in M_n by a set of cartesian coordinates $U^L(x)$ $(..,L,..=0,..,n-1)$. So, the isometric condition assumes the following form:

$$g_{\mu\nu}=\eta_{MN}U^M{}_{,\mu}U^N{}_{,\nu} \qquad (1)$$

where η_{MN} is the metric of M_n; the vectors $U_{,\lambda}$ of the tangent plane to S_4 act as generalized vierbeins[33].

General theorems for the local embedding of S_4 in M_n establish some relations between the properties of the functions $U^L(x)$ and the dimensions of M_n[32]. In particular, if the dimension of M_n are ten, the functions $U^L(x)$ exist and they turn out to be analytic[32,34]. So, the embedding for the 10-dimensional heterotic model guarantees very strong proprieties for the functions $U^L(x)$.

In the neighbourhood of the embedding point $x^\lambda\in S_4$, we then construct $n-4$ vectors $N_m(x)$ $(..,m,..=4,..,n-1)$, with cartesian coordinates $N_m{}^L$, orthogonal to each other and to S_4. The following relations will hold:

$$\eta_{LM}N_m{}^LN_n{}^M=g_{mn} \ ; \quad \eta_{LM}N_m{}^LU^M{}_{,\mu}=0 \qquad (2)$$

where g_{mn} locally represents the metric of the internal space.

Let us now consider a point z^L of M_n which does not necessarily belong to S_4: its embedding coordinates can be expressed by:

$$z^L=U^L(x)+y^mN_m{}^L(x) \qquad (3)$$

The $n-4$ parameters y^m so introduced are of such kind that the previous expression represents in M_n the local relation that allows to move from the coordinates system $\{z^L\}$ to the system $\{x^\lambda,y^l\}$ in which S_4 is defined by the condition $y^l=0$. In the following discussions we will assume y^l as periodic coordinates.

Now we are in the position to write down the multidimensional metric[34]: from $ds_{(n)}{}^2=\eta_{MN}dz^Mdz^N$ and with the help of (1), (2) and (3), we obtain

$$ds_{(n)}{}^2=g_{\mu\nu}dx^\mu dx^\nu+g_{mn}dy^mdy^n+ +\eta_{LM}[2N_m{}^Ldy^m+(2U^L+y^mN_m{}^L)_{,\mu}dx^\mu]y^nN_n{}^M{}_{,\nu}dx^\nu \qquad (4)$$

This expression can be simplified by focusing our attention on an internal space not depending from the external coordinates unless with respect to a scale factor: so, assuming

$$N_m{}^L(x)=e^{\phi(x)}Q_m{}^L \qquad (5)$$

we obtain

$$ds_{(n)}{}^2=g_{\mu\nu}dx^\mu dx^\nu+e^{2\phi}h_{mn}dy^mdy^n+ +e^{2\phi}h_{mn}[y^m\phi_{,\mu}dx^\mu+2dy^m]y^n\phi_{,\nu}dx^\nu \qquad (6)$$

where $h_{mn}=Q_m{}^MQ_n{}^N\eta_{MN}$ is the metric tensor of the internal space not depending from the coordinates $x^\lambda$of the embedding point. Introducing now the scaling coordinates $Y^l=e^\phi y^l$ it turns out to be possible to reconsider (6) in the more compact form

$$ds_{(n)}{}^2=g_{\mu\nu}dx^\mu dx^\nu+h_{mn}dY^mdY^n \qquad (6')$$

The hypothesis (5) allows to come closer to the geometric position contained in the above-mentioned generalization of the Friedmann metric and this permits us to make an immediate comparison: named $g^{(im.)}{}_{MN}$ and $g^{(Fr.)}{}_{MN}$ as, respectively, the multidimensional embedding metric tensor defined by (6) and the Friedmann metric, we have

$$g^{(im.)}{}_{MN}=g^{(Fr.)}{}_{MN}+e^{2\phi}g'_{MN} \qquad (7)$$

where the g'_{MN} blocks are given by

$$g'_{\mu\nu}=h_{mn}y^my^n\phi_{,\mu}\phi_{,\nu} \ ; \quad g'_{m\mu}=h_{mn}y^n\phi_{,\mu} \ ; \quad g'_{mn}=0 \ . \qquad (8)$$

Thus, the two metrics differ for the terms in (8) that explicitly depend from the internal coordinates and from the first derivatives of the scale field of the internal space. This means that the Friedmann metric is equal to the embedding metric only for an observer located in S_4 or, in general, when the scale of the internal space turns out to be constant. In any case, *not the real multidimensional geometry but only the observable geometry on the S_4 hypersurface is exactly of the Friedmann type*.

From the variational principle $\delta\int ds_{(n)}=0$ let us now obtain the geodesic equations relating to the metric (6). Said equations take the following form:

$$(du^\lambda/ds_{(n)})+\Gamma^\lambda{}_{\mu\nu}u^\mu u^\nu+\tfrac{1}{2}\phi^{,\lambda}h_{mn,r}y^rV^mV^n=0 \qquad (9)$$

$$(dV^l/ds_{(n)})+\Gamma^l{}_{mn}V^mV^n-\phi_{,\mu}h^{lp}h_{pm,n}y^nu^\mu V^m=0 \qquad (9')$$

where $u^\lambda=dx^\lambda/ds_{(n)}$, $V^l=dY^l/ds_{(n)}$ and the $\Gamma^L{}_{MN}$ are the usual affine connections. On the contrary, making use of the generalized Fridmann metric we obtain the equations[11]

$$(du^\lambda/ds_{(n)})+\Gamma^\lambda{}_{\mu\nu}u^\mu u^\nu-e^{2\phi}\phi^{,\lambda}h_{mn}v^mv^n=0 \qquad (10)$$

$$(dv^l/ds_{(n)})+\Gamma^l{}_{mn}v^mv^n+2\phi_{,\mu}u^\mu v^l=0 \qquad (10')$$

where $v^l=dy^l/ds_{(n)}$.

Comparing the expressions (9) and (9') with (10) and (10') it follows that these equations differ, to first order in y, for a term proportional to $2h_{mn}+h_{mn,r}y^r$. This term turns out to be the Lie derivative of h_{mn} with respect to the scale transformation $y^m\rightarrow e^\varepsilon y^m$: the latter is not an isometry for the internal space so the previous term cannot be zero.

The most delicate problem of the above scheme consists in the choice of h_{mn}. If we consider, for example, a flat internal space, the equations (9) and (9') are not influenced by the behaviour of the scale field of the internal space: it means that whichever multidimensional model we may consider, the fields equations trivially take back the form of the classic quadridimensional case, so to lose any possibility to follow the dynamic evolution of the extradimensional sector of the universe. On the other hand, the unification model to which this work is aimed provides an internal space of the Kalabi-Yau type[28,30,35] for which $h^{mn}R_{manb}=0$ but $R_{manb}\neq 0$; by electing this choice the equations (9) and (9') do not generate trivial models: this permits to the equations of the g_{MN} field to describe, in addition to the dynamic of $g_{\mu\nu}$, also the dynamic of the ϕ field.

Before ending this section, let us show the components of the Riemann tensor; making use of (9) and (9') we obtain:

$$R_{\alpha\beta\gamma\delta}=R_{(4)\alpha\beta\gamma\delta}\ ;\ R_{abcd}=R_{(D)abcd}\ ;\ R_{\alpha\beta ab}=0\ ;$$
$$R_{\mu abc}=\tfrac{1}{2}\phi_{,\mu}h_{a[c,b]}\ ;\ R_{m\alpha\beta\gamma}=0\ ;\ R_{\alpha a\beta b}=O_{\alpha a\beta b}(y\times\nabla\phi)$$

where $D=n-4$ and the last components explicitly depend on the internal coordinates.

The model. Let us consider the lagrangian density:

$$L=-R+\psi^{-1}(F_{AB}F^{AB}-L^{(G.B.)})+2\psi^{-2}(\psi_{,M}\psi^{,M}) \tag{11}$$

In this expression, any quantity is defined in a 10-dimensional space-time: R is the scalar curvature, ψ the dilaton and $L^{(G.B.)}=R^2-4R_{AB}R^{AB}+R_{ABCD}R^{ABCD}$ the Gauss-Bonnet term; moreover F_{AB} describes the gauge field. All constants have been eliminated by an appropriate redefinition of the ψ field and the L density. The (11) comes from the bosonic sector of the superstrings heterotic model[21,36]. The quadratic Gauss-Bonnet term, which is present for supersimmetry reasons[37,38], does not introduce anomaly in the gravitational propagator[19]. Moreover, it turns out to be consistent with the multidimensional generalization of the gravitational lagrangian density[20,39].

Einstein-Gauss-Bonnet cosmological models have been taken into consideration by several authors[2,8,9,11,12,38,40÷44]. These models, unlike the linear case[45], does not provide generally stable solutions for the system[11,12] and this fact has brought someone to criticize the presence of the Gauss-Bonnet term[12]; in practical, this models introduce the geometric scale field interacting with gravity through the generalization of the Friedmann metric, which does not seem to be supported, as above pointed out, by strong principles. Therefore, it is not clear why we are not to be doubtful about the metric ansatz, rather than the quadratic term; the latter, on the contrary, with respect to unification theories, is supported by strong arguments.

As far as we are concerned, the present work lies on these conjectures.

Let us obtain the field equations from the variational principle $\delta\int(L+L^{(m)})(-g)^{1/2}d^{10}z=0$. In this expression we have introduced the $L^{(m)}$ term that takes into account for the eventual presence of a matter fluid. So we have:

$$G_{MN}=T^{(m)}_{MN}+2\psi^{-2}\psi_{,M}\psi_{,N}-\psi^{-2}(\psi_{,L}\psi^{,L})g_{MN}+$$
$$+\psi^{-1}(2F_{ML}F^{L}{}_{N}-\tfrac{1}{2}F^2g_{MN}+T^{(G.B.)}{}_{MN})+O_{MN}(R\times\nabla\psi) \tag{12}$$

$$\psi_{;L}{}^{L}-\psi^{-1}\psi_{,L}\psi^{,L}=\tfrac{1}{4}(L^{(G.B.)}-F^2) \tag{13}$$

$$F^{AB}{}_{;B}=0 \tag{14}$$

where, in (12), the O_{MN} term contains products beetwen the curvature and the dilaton derivatives and where $T^{(G.B.)}{}_{MN}=\tfrac{1}{2}g_{MN}L^{(G.B.)}-2RR_{MN}+4R_{ML}R^{L}{}_{N}+4R^{AB}R_{MANB}-2R_{MABC}R_{N}{}^{ABC}$.

Let us now consider, in this short exposition, the case in which $\psi^{-1}=\alpha=const$[2,8,9,11,12,38,40÷43]. Besides, let us choose to embed the gauge field in the internal space making use[9,35,41,44] of $F_{A\alpha}=0$ and $F_{ab}F^{ab}=R_{abcd}R^{abcd}$. Introducing in (12) the embedding metric given by (6) and choosing for the stress-energy tensor of the matter field the diagonal expression $T^{(m)}{}_{M}{}^{N}=diag\{\rho,..-p_e\ .,..-p_i\ .\}$, we obtain the equations:

$$G_{\mu\nu}=T^{(m)}{}_{\mu\nu}+\tfrac{1}{2}\alpha\omega^2(\phi_{,\lambda}\phi^{,\lambda}g_{\mu\nu}-\phi_{,\mu}\phi_{,\nu})+O_{\mu\nu}(y^2\times\nabla\phi) \tag{15}$$

$$\tfrac{1}{2}\alpha\omega^2\phi_{,\lambda}\phi^{,\lambda}+(R_4+\alpha L_4{}^{(G.B.)})=2p_i \tag{16}$$

where $G_{\mu\nu}$, R_4, $L_4{}^{(G.B.)}$ are, respectively, the Einstein tensor, the scalar curvature and the Gauss-Bonnet term related to the $g_{\mu\nu}$ metric of S_4. The $O_{\mu\nu}$ term of (15) esplicitly depends from quantities which can be neglected in a low energy model and, anyhow, it will be exacly equal to zero on the S_4 hypersurface.

In (15) and (16) a mass parameter $\omega^2=(h_{a[b,c]})^2$ appears: it does not seem possible to develop it, since any explicit expression for the metric tensor of a Kalabi-Yau space it is still unknown. However, it turns out to be different from zero because of the antisimmetry properties of Riemann tensor indices. In the following discussion, after expanding this quantity in harmonic series on the y^m, we will choose to absorb the zero order term ($\omega_0{}^2$) in the definition of the scale field in the assumption that this position does not change the qualitative behavior of the model.

Let us now have the ordinary space-time geometry consistent with the Cosmological Principle and the scale field spatially omogeneous:

$$ds_{(4)}{}^2=g_{\mu\nu}dx^{\mu}dx^{\nu}=dt^2-a^2(t)ds_{(3)}{}^2 \tag{17}$$

$$\phi=\phi(t) \tag{17'}$$

In (17) we will also suppose $ds_{(3)}{}^2=\Sigma dx^2$. Introducting these hypotesis in (15) and (16) the following equations are obtained:

$$3H^2=\rho \tag{18}$$

$$\alpha\dot{\varphi}^2=3[\dot{H}+(\gamma_i+2)H^2]-12\alpha H^2(\dot{H}+H^2) \tag{19}$$

where $H=\dot{a}/a$ and $\varphi=\tfrac{1}{2}\omega_0\phi$. To them the conservation equation $T_0{}^{\lambda}{}_{;\lambda}=0$ must be added:

$$\dot{\rho}+3(\gamma_e+1)H\rho=6H\alpha\dot{\varphi}^2 \tag{20}$$

In the previous relations we made explicit reference to the equations of state $p_i=\gamma_i\rho$ e $p_e=\gamma_e\rho$.

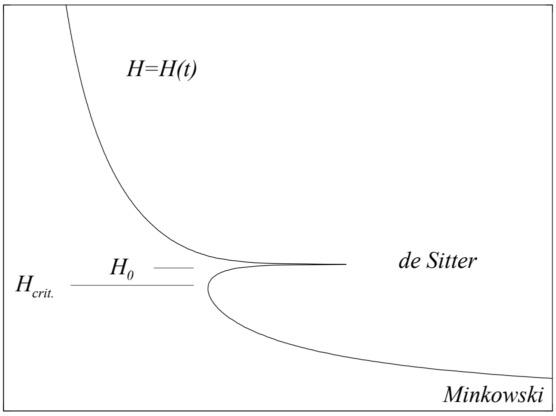
Defining, as a matter of convenience, $\gamma=(3-\gamma_e+2\gamma_i)/8\alpha$, the differential equation for $H(t)$ assumes the form:

$$12H[\dot{H}-6\alpha H^2(\dot{H}+H^2-\gamma)]=0 \qquad (21)$$

The previous equation admits, as fundamental solution, the Minkowski space-time for which it turns out to be: $a=a_0$, $\rho=0$ and $\phi=\phi_0$. Besides, (21) provides for a further solution simply obtained in the inverse form $t=t(H)$. Said solution turns out to be of particular cosmological interest if we assume that the state parameters, reasonably, satisfy the relation $\gamma_e-2\gamma_i<5/3$. We have:

$$t=t_0+\frac{1}{6\alpha\gamma H}+\gamma_0\ln\left|\frac{\gamma^{1/2}+H}{\gamma^{1/2}-H}\right| \qquad (22)$$

where $\gamma_0=(6\alpha\gamma-1)/12\alpha\gamma^{3/2}$ turns out to be, for the choice on the state parameters, a quantity always positive. The qualitative graphic of (22) is showed as follows.



Consequently, the model admits two different asyntotic solutions which control the $H(t)$ evolution when its initial value H_{in} is known: defining $H_{crit.}=(6\alpha)^{-1/2}$ it happens that, if $H_{in}<H_{crit.}$, then $H\rightarrow 0$ (Minkowski) while, if $H_{in}>H_{crit.}$, we have $H\rightarrow H_0=\gamma^{1/2}$ (de Sitter). It is worth mentioning that $H_{crit.}$ does not depend from state parameters.

When $H<<\gamma^{1/2}$, the (22) can be immediately inverted so to obtain $H=[6\alpha\gamma t]^{-1}$ that is $a=a_0 t^{4/3(3-\gamma_e+2\gamma_i)}$: when the ordinary space is radiation dominated ($\gamma_e=1/3$) and the internal space is empty ($\gamma_i=0$) the typical cosmological solutions $a=a_0 t^{1/2}$ and $\rho=\rho_0 t^{-2}$ are obtained. During this phase, the behaviour of the scale field of the internal space is controlled by the following equation:

$$\dot{\varphi}^2=-12\frac{\ddot{a}}{a}\left(\frac{\dot{a}}{a}\right)^2$$

The second member is always positive and it goes to zero as t^{-4} so that $|\phi-\phi_0|\sim t^{-1}$.

The behaviour of the scale field near the de Sitter phase turns out to be given by $|\phi-\phi_0|\sim e^{-t/2\gamma_0}$ if we assume, as usual for an inflationary phase, $\gamma_e=-1$.

What we have discussed before it has to be considered as a toy model: in our system, in fact, we have assumed a strictly constant configuration for the dilaton ψ excluding, as a matter of fact, the possibility that the two scalar fields ϕ and ψ could reach together their final configuration. Besides, the righthand side of (12) makes plausible the hypotesis that it is the dilaton dynamic which controls the inflationary phase of the universe[41]. Moreover, this fact turns out to be more evident if we observe that $H_{crit.}$ and H_0 depend from $\alpha=\psi^{-1}$. The study of the dilaton dynamic will be the object of further researches. In the present work, we wished to solve the non-linear Einstein-Gauss-Bonnet cosmological model without using any ansatz on the geometry but making use of the embedding metric: the obtained results, we believe, give enough credit to this geometric outline.

Conclusions. In the previous section we have seen how the embedding metric given by (6) is able to solve the non-linear Einstein-Gauss-Bonnet cosmological model. *The geometry of the multidimensional universe admits a vacuum configuration given by the direct product* $M_4\times K_6$. *Moreover, for low energy observers the multidimensional universe appears in any case as a topological structure given by a direct product of two subspaces. Finally, in the presence of a matter fluid, the model generates the cosmological solutions tipically contained in the standard 4-dimensional theory.*

The embedding geometry, generally given by the equation (4), seems to reflect a cosmological principle if it is characterized by the hypotesis (5), (17) and (17'): thus, in the following discussion we will try to make the preceding assertion plausible.

As it is known[1], the Cosmological Principle states that the universe is homogeneous and isotropic and this represents a statement about the existence of equivalent coordinates systems. Consistently with the spirit of General Relativity, this corresponds to state that through each event of the space-time there passes a tridimensional spacelike hypersurface on which the density and the curvature turn out to be constant and that the world lines of the cosmological fluid are orthogonal to said hypersurface[46]. This principle, although not containing by itself any denial about the existence of a multidimensional structure of the universe, can be regarded as the geometric principle related to the actually observable sector of the universe. In this way, the ordinary universe must be embedded in the multidimensional one and said embedding must locally characterize an internal geometric structure. The geometric hypotesis contained in (5), (6), (17) and (17') thus affirm that *each event of the ordinary space-time locates a spacelike hypersurface of* *homogeneity and isotropy embedded in a multidimensional space-time and defined in such way that even the scale field of the internal space turns out to be homogeneous on it. Moreover, the topological structure of the internal space is homogeneus on each hypersurface and globally invariant on all the family of the tridimensional hypersurfaces that describe the evolution of the universe.*

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